+6 Analysis of Pictures Taken with an Underwater Camera

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Dit is een ingekorte versie van een artikel in *The Physics Teacher*.

In the Dutch high school system students are required to carry out a research project on a subject of their own choosing. During the section on optics, the teacher (Hubert) mentioned that a fish sees the world above the water in a cone with half-angle equal to the critical angle for the air/water interface (49°). This follows from Snell's law and has been discussed in some detail by Jearl Walker[1]. He describes the intersection of this cone with the surface of the water as the 'window' through which the fish sees the outside world. A related paper has appeared in this journal[2]. Stimulated by the teacher's remark, three students (Nienke, Floor-Jolijn, and Rose) made photographs with an underwater camera in the local swimming pool in Hoorn.

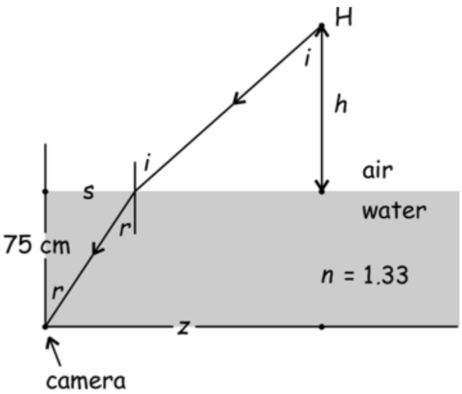
In their own words: 'Initially, the goal was to photograph the entire above-water field of view that would be seen by a fish. However, the acceptance angle of our camera was less than 30°, which is much smaller than the 98° that would have been required. We therefore tried a different experiment. Instead of aiming the underwater camera directly upward, we aimed it at the critical angle. First we tried to photograph objects floating on the surface of the water but were unsuccessful, and so we decided instead to take pictures of a student standing in the water. Near the end of this paper, we will explain why our attempts to photograph the floating objects were not successful. Many of our results were baffling at first and gave Hubert the opportuniry to teach some mathematical concepts in explaining the photographs.'

This photograph shows the setup: Nienke (with snorkel) takes the pictures while Floor-Jolijn holds her underwater and Rose stands a distance z away from the camera.

This is the resulting photograph for z = 150 cm. We placed a ribbon at the bottom of the pool to serve as a coordinate line. The depth of the water was 75 cm. To explain the picture, we show in the following section how to predict where the image locations are, as seen by the camera, for various points on Rose's body.

Location of the Virtual Images

The next figure shows the ray of light from point H (height h above the water) that reaches the camera. The angle of incidence for that ray is i, the angle of refraction is r, and n is the refractive index (1.33 for water). From this figure it follows that for z = 150 cm, $s = 75 \cdot \tan r$ and $150 - s = h \cdot \tan i$ with $0 \le h \le 93$ cm. (Rose is 168 cm tall.)



Now using $\tan\alpha = \dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$ and Snell's law ($\sin i = n \cdot \sin r$), and letting $\sin r = p$ we find:

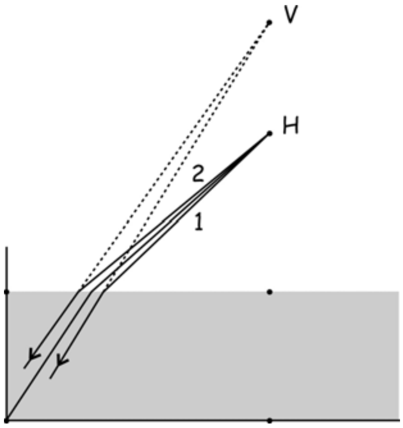
$$150 - \frac{75p}{\sqrt{1-p^2}} - \frac{hnp}{\sqrt{1-n^2p^2}} = 0$$

This equation can be solved numerically for p using, for example, the 'solver' function of a graphing calculator such as the Casio CFX. Once p is known, we may write $r = \sin^{-1}(p)$ and $i = \sin^{-1}(np)$.

Now that we know the path of the ray from point H to the camera, we can compute the paths of two adjacent rays in order to find the location of the virtual image (see next figure). Let these two rays have angles of incidence $i_1 = i - \Delta i$ and $i_2 = i + \Delta i$ (we used $\Delta i = 0.001$ rad in our calculations).

The corresponding angles of refraction are

$$r_1 = \sin^{-1}\left(\frac{\sin i_1}{n}\right) \quad \text{and} \quad r_2 = \sin^{-1}\left(\frac{\sin i_2}{n}\right).$$



The location V of the virtual image of point H is the intersection of the two dotted lines in the figure.

The equations of these lines are $y = 75 + m_1 \cdot (x - s_1)$ and $y = 75 + m_2 \cdot (x - s_2)$, where m_1 and m_2 are the slopes of the two lines:

$$m_1 = \frac{1}{\tan r_1} \quad \text{and} \quad m_2 = \frac{1}{\tan r_2}.$$

The distances s_1 and s_2 are:

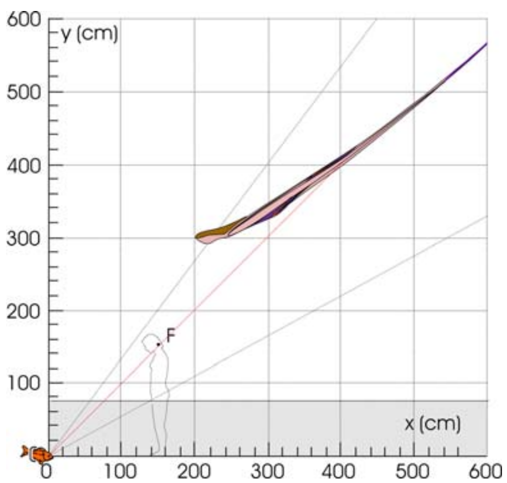
$s_1 = z - h \cdot \tan i_1$ and $s_2 = z - h \cdot \tan i_2$.

The coordinates (x_V, y_V) of the point of intersection are given by:

$$x_V = \frac{m_1 s_1 - m_2 s_2}{m_1 - m_2} \quad \text{and} \quad y_V = 75 + m_1 \cdot (x_V - s_1)$$

A point on Rose's body that is a horizontal distance z from the camera and a vertical distance h above the water will be imaged at the point (x_V, y_V).

By repeating the above calculations (using a simple BASIC or spreadsheet computer program) for a large number of points on Rose's body, her complete image may be found:

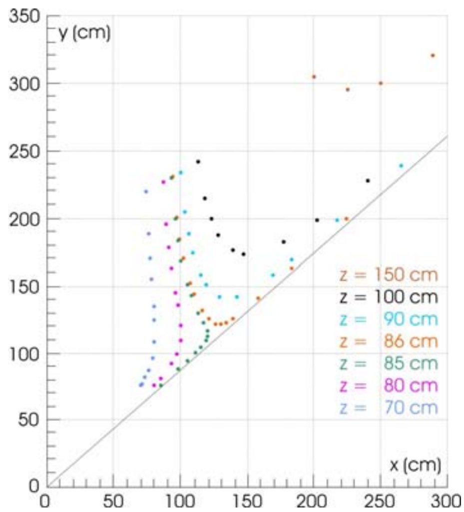


The black lines in this figure give the field of view of the camera; point F is the location of the virtual image of Rose's feet formed by reflection from the surface of the water. With the help of the red line in the figure, we can see why the feet appear near the center of the body in the second photograph.

Values of *h* close to zero

A surprising (to us) result of the computations is that for values of h close to zero, images are formed at very large distances where they cannot be focused sharply by the camera. This explains why we were unsuccessful in our attempts to obtain pictures of objects floating on the surface of the water.

Finally, in the next picture, we show image shapes for several values of z ranging from 70 cm to 150 cm. A linear, 'one-dimensional Rose' standing vertically at a particular location z is imaged into the corresp-onding curve in the figure. The curve for z = 150 cm can readily be compared to the image shape seen in the former picture.



The highest point on each of the other curves represents the image location of the top of Rose's head (h = 93 cm); the remainder of each curve shows the locations of images of successively lower points on her body down to h = 0 (surface of the water).

The results are particularly interesting in the region of $z = z_c = 75 \cdot \tan r_c$ where r_c is the critical angle for the air/water interface.

With $r_c = 48.75°$, we have z_c = 85.5 cm. For $z < z_c$ the image of a point located at h = 0 lies on the surface of the water. For $z > z_c$, as h approaches 0, the image curve asymptotically approaches infinity; the asymptote passes through the origin and has a slope equal to $1/\tan r_c$.

1. Jearl Walker, 'What is a fish's view of a fisherman and the fly he has cast on the water?' *Sci. Am.* **250**, 138 (March 1984).
2. Samuel Derman, 'How the world looks underwater: A demonstration for nonswimmers.' *Phys. Teach.* **20**, 474 (Oct. 1982).